Compact Operators

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Introduction

Linear algebra tells us a great deal about the properties of operators between finite dimensional spaces, and about their specturm.

In general, the situation is considerably more complicated in infinite dimensional spaces. However, there is a class of operators in infinite dimensions for which a great deal of the finite dimensional theory remains valid. This is the class of compact operators.

Their spectral theory is much simpler than that of general bounded operators, and it is just a bit more complicated than that of bounded finite rank operators.

Introduction

Compact operators are important not only for the well-developed theory which is available for them, but also because compact operators are encountered in very many important applications.

Many problems in mathematical physics lead to integral equations, and the associated integral operators are compact.

In the lecture, we discuss few properties and results on compact operators.

Compact Operators

Let X, Y be normed spaces.

A linear operator K from X to Y is called **compact** if $D(K) = X$ and for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \leq c$, the sequence $\{Kx_n\}$ has a subsequence which converges in Y.

Equivalently, K is compact if it maps bounded sequences in X to sequences in Y which have convergent subsequences.

The set of all compact operators from X to Y is denoted by $K(X, Y)$.

If $X = Y$, we write $K(X)$ instead of $K(X, X)$.

Compact and Relatively Compact Sets

Definition 1.

A subset S of a normed space X is **compact** *if every sequence of elements in S has a subsequence which converges to an element of S.*

Definition 2.

A subset S of a normed space X is said to be **relatively compact** *if every sequence of elements of S has a convergent subsequence* **converging to an element of** X.

The limit of this subsequence, however, need not be in S.

The crucial difference between definitions of compact and relatively compact is that the limit is not required to belong to the set S.

Compact and Relatively Compact Sets

Definition of"relatively compact" is obviously weaker than the definition of "compact." Every compact set is relatively compact but not vice versa.

An open interval in $\mathbb{R}$ is the standard example of a relatively compact set which is not compact.

If X is a finite dimensional normed space, then every bounded set is relatively compact (every finite dimensional normed space X has a Bolzano Weierstrass property : each bounded sequence in X has a convergent subsequence). The unit ball in an infinite dimensional normed space X is not relatively compact. Hence the identity map I on X is linear and continuous but it is not compact.

Exercise 3.

Show that an orthonormal sequence $\{e_n\}$ in a Hilbert space H cannot have a convergent subsequence.

Proposition 4.

Let S be a subset of a normed space X. Then S is relatively compact if and only if $\overline{S}$ is compact.

Hence every relatively compact set is a set with compact closure.

Proposition 5.

Every subset of a compact or relatively compact set is relatively compact.

Proposition 6.

Let X, Y be normed spaces. A linear operator $K : X \to Y$ (defined everywhere) is compact if and only if $K(B)$ is relatively compact, for any bounded set $B \subset X$. FA-1(P-32)P-1

Proposition (6) says that K is compact if and only if image of every bounded subset of X is a relatively compact subset of Y.

Proposition 7.

Every compact operator is bounded. That is, $K(X, Y) \subseteq B(X, Y)$.

FA-1(P-32)P-2

Proposition 8.

$K(X, Y)$ is a subspace of $B(X, Y)$. FA-1(P-33)P-3

Exercise 9.

Let X, Y be normed spaces. Let $U(0,1) = \{x \in X : \|x\| < 1\}$ and $B(0,1) = \{x \in X : \|x\| \leq 1\}$ be the open and closed unit ball in X respectively. Let $K : X \to Y$ be a linear operator and $r > 0$ be any real number. Prove that the following statements are equivalent :

1. *K is compact ;*
2. *$K(U)$ is relatively compact subset of Y.*
3. *$K(B)$ is relatively compact subset of Y.*
4. *$K(U(0,r))$ is relatively compact subset of Y.*
5. *$K(B(0,r))$ is relatively compact subset of Y.*

Compact Operators

We have seen that a linear operator T from a normed space X to a normed space Y is continuous if and only if it sends the open unit ball U in X to a bounded subset of Y. Then the closure of $K(U)$ is a closed and bounded subset of Y.

Compact operator sends the open unit ball U in X to a relatively bounded subset of Y. Then the closure of $K(U)$ is a compact subset of Y.

Finite Rank Operators

Definition 10.

Let X, Y be normed spaces. A linear operator $T : X \to Y$ is said to be an **operator of finite rank** *if the range of T is of finite dimensional.*

The set of all **bounded linear finite rank** of operators is denoted by $BFR(X, Y)$. Note that every finite rank linear operator is not necessarily bounded.

Theorem 11.

If $A \in B(X, Y)$ and $K \in BFR(Y, Z)$, then $KA \in BFR(X, Z)$. Similarly, if $L \in BFR(X, Y)$ and $B \in B(Y, Z)$, then $BL \in BFR(X, Z)$.

That is, the product of a bounded operator and an operator of finite rank (in either order) is an operator of finite rank.

Theorem 12.

Every bounded finite rank operator is compact. FA-1(P-33)P-4

Corollary 13.

If either X or Y is finite dimensional, then $B(X, Y) = K(X, Y)$. FA-1(P-34)C-5

In particular, a linear functional on a normed space is compact if and only if it is bounded. That is, $K(X, \mathbb{K}) = X^*$.

Exercises 14.

(a) *Suppose that X is a reflexive Banach space. Prove that every member of $B(X, \ell_1)$ is compact.*

(b) *Suppose that Y is a reflexive Banach space. Prove that every member of $B(c_0, Y)$ is compact.*

Examples

In infinite dimensions, there are many operators which are not compact. In fact, compactness is significantly stronger property than bounded.

Example 15.

For $1 \leq p \leq \infty$, the **right shift operator** $R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ *and* **left shift operator** $L(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ *from ℓ_p to ℓ_p,* **are not compact.**

As for all $n, m \in \mathbb{N}, n \neq m$ $\|e_n - e_m\|_p = \begin{cases} 2^{1/p} & \text{if } 1 \leq p < \infty \\ 1 & \text{if } p = \infty, \end{cases}$

the image of the bounded sequence $\{e_n\}$ under R is $\{e_{n+1}\}$, which has no convergent subsequence.

Also, the image of the bounded sequence $\{e_n\}_{n=2}^{\infty}$ under L is $\{e_n\}_{n=1}^{\infty}$, which has no convergent subsequence.

Exercise 16.

Let $\{\lambda_n\}$ *be a sequence of scalars such that* $\lambda_n \to \lambda \neq 0$.

For $1 \leq p \leq \infty$, *let* $A : \ell_p \to \ell_p$ *be the (diagonal) operator associated with this sequence, defined by*

$$A(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots), \{x_n\} \in \ell_p.$$

Then, prove that $A \in B(\ell_p)$, $\|A\| = \sup_n |\lambda_n|$ *and* $\{Ae_n\}$ *does not have any convergent subsequence, and hence* A **is not a compact operator.**

FA-1(P-34)E-6

Proposition 17.

The identity operator of a normed space is compact if and only if the space is of finite dimension.

Proposition 18.

X_0 is a subspace of a normed space X. The inclusion operator $I_0 : X_0 \to X$ is compact if and only if X_0 is finite dimensional.

Theorem 19.

If $A \in B(X, Y)$ and $K \in K(Y, Z)$, then $KA \in K(X, Z)$. Similarly, if $L \in K(X, Y)$ and $B \in B(Y, Z)$, then $BL \in K(X, Z)$.

That is, the product of a bounded operator and a compact operator (in either order) is a compact operator. FA-1(P-35)T-7

Corollary 20.

A compact operator of an infinite dimensional normed space is not invertible in $B(X)$. FA-1(P-65)C-32

Exercises 21.

(a) *Show that the range of compact operator from a Banach space into a Banach space is closed if and only if the operator has finite rank.*

FA-1(P-67)E-34

(b) *Prove that bounded operators between normed spaces map relatively compact sets to relatively compact sets.*

(c) *If F is continuous and of finite rank, then F is a compact operator and $R(F)$ is closed in Y. Conversely, if X and Y are Banach space, $F : X \to Y$ is a compact operator and $R(F)$ is closed in Y, then F is continuous and of finite rank.*

FA-1(P-69)E-35

Proposition 22.

Let M be a closed subspace of a Hilbert space H. Let $K : H \to M$ be the orthogonal projection onto M. Then K is compact if and only if M is finite dimensional. FA-1(P-42)P-12

Proposition 23.

Let H be a Hilbert space and let $T : H \to H$ be an idempotent operator. Then T is compact if and only if it is of finite rank. FA-1(P-42)P-13

Exercise 24.

Let A and B be bounded operators on an infinite dimensional Hilbert space H. Which of the following are true?

(a) *If AB is compact, then either A or B is compact?*

(b) *If $A^2 = 0$, then A is a compact operator.*

(c) *If $A^n = I$, for some $n \in \mathbb{N}$, then A is not compact.*

Definition 25.

For an arbitrary ring $(R, +, .)$ *(two operations, called "addition" and "multiplication"), let* $(R, +)$ *be its additive group.*

A subset I *is called a* **two-sided ideal** *(or simply an* **ideal***) of* R *if it is an additive subgroup of* R *that "absorbs multiplication by elements of* R*."*

Formally, we mean that I *is an ideal if it satisfies the following conditions :*

1. $(I, +)$ *is a subgroup of* $(R, +)$ *;*
2. $x.r,\ r.x \in I$, *for all* $x \in I, r \in R$.

"Two-sided" refers to the fact that we may multiply by any element of R *from either side.*

The set $K(X)$ of compact operators forms a two-sided ideal.

From Proposition (8) and Theorem (19), we have the following.

Proposition 26.

$K(X)$ is a two-sided ideal of the normed algebra $B(X)$. If X is a Banach space, then $K(X)$ is a two-sided ideal of the Banach algebra $B(X)$.

The multiplication of two bounded operators in $B(X)$ is the composition of them. The study of Banach algebras has been a major theme in functional analysis. We shall discuss Banach algebras later.

Theorem 27.

Let X be a normed space and Y a Banach space. If L is in $B(X, Y)$ and there is a sequence $\{K_n\} \subset K(X, Y)$ such that

$$\|L - K_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

then L is in $K(X, Y)$. FA-1(P-35)T-8

The above Theorem (27) tells that $K(X, Y)$ **is a closed subspace of $B(X, Y)$.** Thus $K(X, Y)$ is a Banach space.

Hence Proposition (26) says that if X is a Banach space, then $K(X)$ is a closed two-sided ideal of the Banach algebra $B(X)$.

Finite Rank Operators

Corollary 28.

Let X be a normed space and Y a Banach space. If L is in $B(X, Y)$ and there is a sequence $\{F_n\}$ of bounded operators of finite rank such that

$$\|L - F_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

then L is in $K(X, Y)$.

Corollary (28) says that the uniform limit of a sequence of finite rank operators is compact.

Exercise 29.

Show that a strong limit of finite rank operators is not always compact.
FA-1(P-67)E-33, FA-1(P-71)E-36

Is the converse of Corollary (28) true?

Is every compact operator the uniform limit of a some sequence of bounded finite rank operators?

Is the converse of Corollary (28) true?

Suppose K is a compact operator. Can we find a sequence of bounded operators of finite rank that converges to K in norm?

In other words, is $BFR(X, Y)$ a dense subspace of $K(X, Y)$?

The answer is yes, when Y is a Hilbert space, using orthogonal projections.

It is also affirmative for many well-known Banach spaces. For many years it was believed to be true for a general Banach space. However, examples of Banach spaces were found for which the assertion is false.

This question has been answered negatively by Enflo, in 1973. There exists a (even separable) Banach space on which some compact operator is not a uniform limit of bounded finite rank operators.

Definition 30.

A Banach space for which the finite rank operators are norm-dense in the compact operators is said to have the **approximation property***.*

For a long time one of the major open problems in Banach space theory, the approximation problem, was whether every Banach space has the approximation property.

In 1932, Banach inquired whether every compact operator on every separable Banach space is a norm limit of sequence of continous operators of finite rank.

The Scottish Cafe

The Scottish Cafe was the cafe in Lwow now Lviv where, in the 1930s and 1940s, Polish mathematicians from the Lwow School of Mathematics met and spent their afternoons discussing mathematical problems. The cafe building now houses the Universalny Bank at 27, Taras Shevchenko Prospekt.

Stanislaw Ulam recounts that the tables of the cafe had marble tops, so they could write in pencil, directly on the table, during their discussions.

To keep the results from being lost, and after becoming annoyed with their writing directly on the table tops, Stefan Banach's wife provided the mathematicians with a large notebook, which was used for writing the problems and answers and eventually became known as the "Scottish Book".

The book - a collection of solved, unsolved, and even probably unsolvable problems - could be borrowed by any of the guests of the cafe.

Solving any of the problems was rewarded with prizes, with the most difficult and challenging problems having expensive prizes during the Great Depression and on the eve of World War II, such as a bottle of fine brandy.

For problem 153, which was later recognized as being closely related to Stefan Banach's "basis problem", Stanislaw Mazur offered the prize of a live goose.

This problem was solved only in 1972 by Per Enflo, who was presented with the live goose in a ceremony that was broadcast throughout Poland. Stanislaw Mazur gives the living goose , the prize for solving probem 153, to Per Enflo in 1972.

* Image taken from wikipedia

Approximation Property

Grothendieck studied the problem intensely in the fifties, trying to prove that every Banach space has the Approximation Property, but he failed. It had been established by Grothendieck that an example of a Banach space lacking the approximation property would also provide a negative answer to a question of Mazur from real analysis. It remained an open question until **Per Enflo** constructed a counterexample in 1973. He found a separable reflexive Banach necessarily infinite dimensional, that lacks the approximation property.

A lot of work has been put into investigating the property, but examples are still not easy to identify.

Approximation Property

On November 6, 1936, Mazur had entered his question in the famous "Scottish book" of open problems kept at the Scottish Coffee House in Lwów, Poland, by Banach, Mazur, Stanislaw Ulam, and other mathematicians in their circle.

Mazur offered a live goose as the prize for a solution. About a year after solving the problem, Enflo travelled to Warsaw to give a lecture on his solution, after which he was awarded the **goose**.

A result in Hilbert spaces contrast with Banach spaces

Theorem 31.

Let X be a normed space, Y be a **Hilbert space** *and $T \in K(X, Y)$. Then there is a sequence of bounded finite rank operators $\{T_n\}$ which converges to T in $B(X, Y)$.*

In other words, the set of bounded finite rank operators is dense in the set $K(X, Y)$ of compact operators.

That is, $\overline{BFR(X, Y)} = K(X, Y)$.

We shall see the proof of the theorem after some examples.

The following exercise shows that in fact, if Y is a Banach space with a Schauder basis, then every $K \in K(X, Y)$ arises in the way.

Exercise 32.

Let X be a normed space and Y be a Banach space. If Y admits a Schauder basis and $K \in K(X, Y)$, then there is a sequence $\{F_n\}$ of finite rank operators in $B(X, Y)$ such that

$$\|F_n - K\| \to 0 \quad as \quad n \to \infty.$$

A result in Hilbert spaces contrast with Banach spaces

The conclusion of the previous theorem is known to be false in Banach spaces, although the only example known to this author (Per Enflo, Acta Math., Vol. 130, 1973) is rather complicated. Certainly the argument using orthogonal projections cannot be employed for Banach spaces.

In 1974, Alexander showed that if $2 < p < \infty$, then there is a compact operator A on a closed subspace of ℓ_p which is not the uniform limit of a sequence of continuous operators of finite rank.

Why are compact operators looked at?

Let X be a normed space and Y be a Hilbert space. Theorem (31) says that an operator $T : X \to Y$ is compact if and only if T is the uniform limit of a sequence of bounded finite rank operators.

This suggests that compact operators are the most direct generalization of matrices of infinite dimensional spaces.

In fact, compact operators retain a very important property of matrices, namely that a compact operator has an empty continuous specturm (a value is in the continuous spectrum of A if it is not an eigenvalue, but the range of $(A - \lambda I)$ is a proper dense subset of the Hilbert space).

Corollary 33.

Let $X = Y = H$ and H be a Hilbert space. Then every compact operator on H is the uniform limit of a sequence of bounded finite rank operators in $B(H)$.

The Corollary (28) gives a process which is one of the most common ways of proving that an operator is compact.

Example 34.

Let $\{\lambda_n\}$ be a bounded sequence of scalars. For $1 \leq p \leq \infty$, the operator $A : \ell_p \to \ell_p$ defined by $A(x_n) = (\lambda_n x_n)$. We have seen that if $\{\lambda_n\}$ converges to a non-zero scalar, then A is not a compact operator.

Now suppose that $\lambda_n \to 0$. For each $k \in \mathbb{N}$, define $A_k : \ell_p \to \ell_p$ by

$$A_k(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_k x_k, 0, \ldots).$$

The operators A_k are bounded operators of finite rank. For each $k \in \mathbb{N}$, $\|(A_k - A)x\|_p \leq (\sup_{i>k} |\lambda_i|)\|x\|_p$, for all $x \in \ell_p$. Hence $\|A_k - A\|_p \leq \sup_{i>k} |\lambda_i| \to 0$ as $n \to \infty$, so $A_k - A$ is bounded, for each k, hence A is bounded. Thus by Corollary (28), A is compact. FA-1(P-51)E-19

Examples of Compact Operators

In each of the following two examples, we consider an operator on sequence spaces induced by an infinite matrix (a_{ij}) of scalars.

That is, on a suitable space X, we define the operator A by

$$A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = (a_{ij})\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}, \quad \text{where } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \in X.$$

Example 35.

Let (a_{ij}) be an infinite matrix of scalars such that $\alpha_j = \sum_{i=1}^{\infty} |a_{ij}|$ (the column sum of absolute values) and $\alpha := \sup_j \alpha_j < \infty$.

We know that $A \in B(\ell_1)$ and $\|A\| = \alpha$. If $\alpha_j \to 0$ as $j \to \infty$, then by Corollary (28), $A \in K(\ell_1)$.

Example 36.

Let (a_{ij}) *be an infinite matrix of scalars such that* $\beta_i = \sum_{j=1}^{\infty} |a_{ij}|$ *(the row sum of absolute values) and* $\beta := \sup_i \beta_i < \infty$.

We know that $A \in B(\ell_\infty)$ *and* $\|A\| = \beta$. *If* $\beta_i \to 0$ *as* $i \to \infty$, *then by Corollary (28),* $A \in K(\ell_\infty)$.

In order to prove the Theorem (31) which states that the set of bounded finite rank operators is dense in the set $K(X, Y)$ of compact operators, we shall develop a few concepts.

Definition 37.

*Let U be a subset of a normed space X. Let $\varepsilon > 0$ be given. A set of points $W \subset X$ is called an ε-**net** for a set $U \subset X$ if for each $x \in U$ there is a $z \in W$ such that $\|x - z\| < \varepsilon$.*

*A subset $U \subset X$ is called **totally bounded** if for every $\varepsilon > 0$, there is a **finite set** of points $W \subset X$ which is an ε-net for U.*

That is, there are points $x_1, x_2, \ldots, x_n$ in X such that

$$U \subseteq \cup_{i=1}^{n} B(x_i, \varepsilon).$$

In the definition of totally bounded set, we actually do not need to have the points x_i's belong to U. But the following proposition says that the points x_i's could be chosen from U itself if the set is known to be totally bounded.

Proposition 38.

Let U be a subset of a normed space X. If U is totally bounded, then for each $\varepsilon > 0$ it has finite ε-net $W \subset U$.

Proposition 39.

Let U be a subset of a normed space X.

1. *If U is relatively compact, then it is totally bounded.* FA-1(P-43)P-14
2. *Converse of (1) : U is totally bounded and X is complete, then U is relatively compact.* FA-1(P-44)P-15
3. *If U is relatively compact, then U is separable.* FA-1(P-37)L-9A

To prove the Theorem (31), the following important result is very useful when Y is a Hilbert space.

Theorem 40.

Let X, Y be normed spaces. If K is a compact operator, then $R(K)$ and $\overline{R(K)}$ are separable. FA-1(P-37)T-9 FA-1(P-60)E-26

The above result says that if K is compact then even if the space X is "big" (not separable) the range of K is "small" (separable).

In a sense, this is the reason why the theory of compact operators has many similarities with that of operators on finite dimensional spaces.

A result in Hilbert spaces contrast with Banach spaces

We shall now prove the following theorem.

Theorem 41.

Let X be a normed space, Y be a **Hilbert space** *and $T \in K(X, Y)$. Then there is a sequence of bounded finite rank operators $\{T_n\}$ which converges to T in $B(X, Y)$.*

In other words, the set of bounded finite rank operators is dense in the set $K(X, Y)$ of compact operators.

That is, $\overline{BFR(X, Y)} = K(X, Y)$. *FA-1(P-38)T-10*

Theorem 42.

Let X, Y be normed spaces, and assume that K is in $K(X, Y)$. Then K^* is in $K(Y^*, X^*)$. *FA-1(P-46)T-16*

Example 43.

Let $F : \ell_p \to \ell_q$ *be defined by*

$$F(x)(i) = \sum_{j=1}^{\infty} k_{ij} x(j), \quad 1 = 1, 2, \quad , x \in \ell_p.$$

(a) *For* $1 < p \leq \infty, 1 \leq q < \infty$, *if* $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^q < \infty$, *then* $F \in K(X, Y)$.

FA-1(P-55)E-23

(b) *For* $p = 1, q = \infty$, *if* $\alpha_i = \sup_{j \geq 1} |k_{ij}| < \infty$, *for each* i, *and* $\alpha_i \to 0$ *as* $i \to \infty$, *then* $F \in K(X, Y)$.

FA-1(P-56)E-23

Exercise 44.

Let $X = \ell_2$ and W be the set of all operators A in $B(X)$ which are given by

$$A(x)(i) = \sum_{j=1}^{\infty} k_{ij} x(j), \quad i = 1, 2, \dots, \quad x \in \ell_2,$$

where $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^2 < \infty$. Show that W is a proper subspace of $K(X)$.

FA-1(P-56)E-24

Exercise 45.

Suppose that X and Y are normed spaces and X is infinite dimensional. Suppose that $F : X \to Y$ is linear and bounded below. Show that F is not compact. Deduce that the identity map on an infinite dimensional normed space is not compact. FA-1(P-48)T-17

Exercise 46.

If X is an infinite dimensional normed space and $F \in K(X)$, then prove that F is not invertible. FA-1(P-50)C-17A

Exercise 47.

Let X be a normed space, $z \in X$ and $f \in X^*$. Show that $T : X \to X$ defined by

$$T(x) = f(x)z, \quad x \in X$$

is a compact operator. FA-1(P-51)E-18

Exercise 48.

Let $X = \ell_p$ where $1 \le p \le \infty$, and $\{\alpha_n\}$ be a sequence of scalars such that $\alpha_n \to 0$ as $n \to \infty$. Define $A : X \to X$ by

$$A(x)(n) = \alpha_n x(n), \quad n = 1, 2, \ldots, \ x \in X.$$

Show that A is compact. FA-1(P-51)E-19

Exercise 49.

If X, Y are Banach spaces and there is a surjective compact operator $T : X \to Y$, then Y is finite dimensional. FA-1(P-52)T-20

Does there exist a compact operator $T : \ell_\infty \to \ell_\infty$ which is surjective?

FA-1(P-53)E-21

Restriction of a compact operator T to a subspace of a domain of T

Exercise 50.

Let X, Y be normed spaces. Prove that the restriction of a compact operator $T : X \to Y$ on a subspace $Z \subseteq X$ is still compact. FA-1(P-53)E-22A

Restriction of non-compact operator to an infinite dimensional subspace could be a compact operator.

If X, Y are normed space, $F \in B(X, Y)$ and F is not compact, then the restriction of F to an infinite dimensional subspace may be compact.

Example 51.

Let $X = \ell_2$. Define $A : X \to X$ by

$$A(x_1, x_2, x_3, \ldots) = (x_1, 0, x_3, 0, \ldots).$$

A is a bounded operator with closed range. Let $Y = R(A)$. Consider $F : X \to Y$. Since F is linear surjective and dim $R(F) = \infty$, F cannot be compact.
Let $Z = span\{e_1, e_2, e_4, e_6, e_8, \ldots\} = \{(x_1, x_2, 0, x_4, 0, x_6, 0, \ldots)\}$. Here dim $Z < \infty$. Let $G = F|_Z$. Then G has finite rank because $R(G) =$ span $\{e_1\}$. Thus G is compact. FA-1(P-53)E-22

Restriction of a compact operator T to a subspace of a co-domain of T.

Exercise 52.

Let X, Y be normed spaces and $F \in K(X, Y)$. Does it follow that

1. $F \in K(X, R(T))$?
2. $F \in K(X, \overline{R(T)})$?

FA-1(P-58)E-25

Exercise 53.

Suppose that X is a normed space, $A \in B(X)$ and $\{x_n\}$ is a bounded sequence in X such that $\{Ax_n - x_n\}$ converges in X. Show that if A^m is compact for some positive integer m, then $\{x_n\}$ has a convergent subsequence. *FA-1(P-62)E-27*

Exercise 54.

Let $X = C[0,1]$ *and let* $A : X \to X$ *be defined by*

$$A(x)(s) = \int_0^1 k(s,t)\ x(t)\ dt, \quad 0 \le s \le 1, x \in X.$$

Then prove that A *is a compact operator when each of the following cases holds true.*

1. k *is a continuous scalar-valued function on the closed unit sphere* $S = \{(s,t) : 0 \le s \le 1,\ 0 \le t \le 1\}$.
2. $k(s,t)$ *satisfies the following two conditions :*
   - (a) $\sup_{0 \le s \le 1} \int_0^1 |k(s,t)|\ dt < \infty$.
   - (b) $\lim_{\delta \to 0} \int_0^1 |k(s+\delta, t) - k(s,t)|\ dt = 0$.

FA-1(P-63)E-28

Exercise 55.

Let $X = L_2[0,1]$ and let $A : X \to X$ be defined by

$$A(x)(s) = \int_0^1 k(s,t)\, x(t)\, dt, \quad 0 \le s \le 1, x \in X.$$

Then prove that A is a compact operator when k holds true in each of the following cases.

1. *k is a continuous scalar-valued function on the closed unit sphere $S = \{(s,t) : 0 \le s \le 1,\ 0 \le t \le 1\}$.*
2. *k is a scalar-valued measurable function on the closed unit square such that $\int_0^1 \int_0^1 |k(s,t)|^2\, ds\, dt < \infty$.*

FA-1(P-64)E-29

Exercise 56.

Let $X = L_2[0,\infty)$. *For* $x \in X$, *let*

$$A(x)(s) = \frac{1}{s}\int_0^s x(t)\,dt, \quad 0 < s < \infty.$$

Prove that $A \in B(X)$ *but* $A \notin K(X)$. FA-1(P-65)E-30

Exercise 57.

Let $x \in \ell_2$ *and* $n = 1, 2, \ldots,$ *let*

$$A(x)(n) = \frac{x(1) + x(2) + \cdots + x(n)}{n}.$$

Prove that $A \in B(\ell_2)$ *but* $A \notin K(\ell_2)$. FA-1(P-65)E-31

Exercises 58.

(a) *Let H be a Hilbert space and let $y, z \in H$. Define $T \in B(H)$ by $Tx = \langle x, y \rangle z$. Show that T is compact.*

(b) *Let H be an infinite dimensional Hilbert space with an orthonormal basis (e_n) and let $T \in B(H)$. Show that if T is compact, then $\lim_{n\to\infty} \|Te_n\| = 0$.* FA-1(P-40)E-11

Conclusion

Though we concentrated on compact operators between arbitrary Banach spaces, the theory of compact operators between Hilbert spaces, especially the theory of the Banach algebra $K(H)$ where H is a Hilbert space, is particularly rich.

In this situation, many of the results can be proved in somewhat different and other simpler ways.

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